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Exterior Differential System Approach to Continuity Equation

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Abstract: **The theory of exterior differential systems is applied to study partial differential equations, expressed in the terms of differential forms using Cartan's method. The continuity equation and the Laplace equation are described by these exterior differential systems involving a geometrical perspective.**

Keywords: **Partial differential equations, exterior derivative, ideal, exterior differential systems, continuity equation.**

I. INTRODUCTION

The theory of exterior differential systems was founded by Elie Cartan [3]. It provides a geometric and coordinate-free approach to the study of differential equations. In analyzing partial differential equations it is more convenient to deal with differential forms which essentially encodes the information of the partial differential equation, than to deal with the partial differential equation itself. Differential forms are the geometric objects which have the properties, invariance under arbitrary mappings and integrands for multidimensional integrals, which make them suitable for description of physical and geometric phenomena. Since physical laws (and many geometric ones too) are described by differential equations, it seems desirable to have a way to describe them in terms of differential forms, this was done by E. Cartan, leading to a theory which he called that of exterior differential systems. We call the system of differential forms that corresponds to the given partial differential equation as exterior differential system.

The latter, however, possess some advantages among which are the facts that the forms themselves often have a geometrical meaning, and that the symmetries of the exterior differential system are larger than those generated simply by changes of independent and dependent variables. Another advantage is that the coordinate free treatment naturally leads to the intrinsic features of many systems of partial differential equations. Exterior differential systems play an important role in mathematics in general as well as in differential geometry since they are a way of studying PDE from a geometric viewpoint.

In this paper, our goal is to use Cartan's theory of exterior differential systems to show that it is indeed possible to rewrite continuity equation in this form. Within this context, our objectives could be stated more precisely as:

a) To explain how to find an appropriate geometric setting for studying a given system of PDE.

b) To show that continuity equation can be reformulated in an intrinsically geometric form.

2. THE CONTINUITY EQUATION

In this section, we introduce brief review of the classical continuity equation in classical space. For more details, see [1].

One of the fundamental equations of fluid mechanics is the continuity equation. This equation is derived from the mass conservation principle, states that the rate of change of the mass of fluid within equal to the net amount of the fluid out of it. In vector notation, the continuity equation is given by the form

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{v}) = 0, \qquad (2.1)$$

Where ρ is the fluid density function ($\rho = \rho(x, y, z, t)$), $\boldsymbol{v}$ is the fluid velocity

($\boldsymbol{v}(x, y, z, t) = \big(u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)\big)$) and t is time. This equation in Cartesian components takes the following form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \qquad (2.2)$$

If variations in density are small compared to the background density the fluid is said to be incompressible (i.e. ρ is constant), the rate of change of ρ following the motion is zero, that is:

$$\frac{D\rho}{Dt} = 0, \qquad (2.3)$$

the continuity equation then takes the simplified form

$$\nabla . \boldsymbol{v} = 0, \qquad (2.4)$$

Or, in Cartesian components,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad (2.5)$$

Which means that the velocity field of an incompressible fluid is divergence free, which in effect says that the volume (rather than the mass) of fluid particles is constant in time. The flow field is also said to be solenoidal under these circumstances.

If, further, the fluid moving irrotationally, then

$$\nabla \times \boldsymbol{v} = 0. \qquad (2.6)$$

This equation guarantees the existence of scalar function φ called the velocity potential such that

$$\boldsymbol{v} = -\nabla \varphi, \qquad (2.7)$$

then the equation of continuity for incompressible fluid (2.4) namely $\nabla . \boldsymbol{v} = 0$ takes the form

$$\nabla^2 \varphi = 0, \qquad (2.8)$$

At all points of the fluid This equation known as Laplace's equation, which in Cartesian components takes the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \qquad (2.9)$$

3. EXTERIOR DIFFERENTIAL SYSTEMS STRUCTURES AND CONCEPTS

In this section, we recall some of the basic ideas of the theory of exterior differential systems. A much more extensive treatment of these concepts can be found in [2].

Let M be a smooth manifold of dimension n. Denote by $\Omega^k(M)$ the space of differential k-forms on M, where $\Omega^0(M) = C^\infty(M)$ is the space of smooth functions on M. Then $\Omega(M) = \Omega^0(M) \oplus \ldots \oplus \Omega^n(M)$ denotes the space of all differential forms on M. There are two operations defined on $\Omega^k(M)$ for all k, an associative and distributive product called the *wedge product* of two differential forms $\wedge: \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$, which satisfies the following properties:

i. For each $f \in \Omega^0(M)$, $f(\omega \wedge \theta) = f\omega \wedge \theta = \omega \wedge f\theta$.

ii. Skew-Commutivity: for $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

(Note that $\alpha \wedge \alpha = 0$ for $\alpha \in \Omega^1(M)$).

Another operator on the space of forms is the exterior derivative, which is defined to be the unique map $\boldsymbol{d}: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ for all k, such that the following properties hold:

i. For each $f \in \Omega^0(M)$, $\boldsymbol{d}f = df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n$.

ii. For $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\boldsymbol{d}(\alpha \wedge \beta) = \boldsymbol{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \boldsymbol{d}\beta$.

iii. For every form α, $\boldsymbol{d}(\boldsymbol{d}\alpha) = 0$.

The space of all forms on M together with the wedge product is called the exterior algebra (graded algebra) on M, which we denote by $\Omega^*(M)$. An element in $\Omega^*(M)$ is homogenous if all its terms have the same degree.

In general, by an exterior differential system we will mean a collection of differential forms $\{\omega^1, \omega^2, \dots\}$ defined on an n-dimensional manifold M.

Definition 3.1: An *ideal* I in the algebra $\Omega^*(M)$ is a sub ring of $\Omega^*(M)$ that is closed under the wedge product with arbitrary differential forms (i.e. for all $\alpha \in I$ implies $\alpha \wedge \beta \in I$ for all $\beta \in \Omega^*(M)$.

Given an exterior ideal I on an m-dimensional manifold M, we let $I^{(k)} = \{\omega \in I | deg\omega = k\}$, $0 \le k \le m$, denote its homogeneous component consisting of all k-forms in I. Thus $I^{(0)}$ will denote the space of all functions (0-forms) in I, while $I^{(1)}$ will denote the space of all one-forms, etc.

Definition 3.2: An ideal I is called homogeneous if the ideal is a direct sum

$$I = \oplus_k I^{(k)}, \tag{3.1}$$

Where $I^{(k)} = I \cap \Omega^k(M)$ is the space of k-form in I (with $I^{(k)} \subset \Omega^k(M)$).

Definition 3.3: A set of differential forms $\{\omega^1, \omega^2, \dots, \omega^r\}$ is said to generate the ideal I if every form $\theta \in I$ can be written as a finite "linear combination" $\theta = \sum_j \eta^j \wedge \omega^j$, where the η^j are arbitrary differential forms satisfying $deg\,\theta = deg\,\eta^j + deg\,\omega^j$.

Definition 3.4: If $S \subset \Omega^*(M)$ is a subset consisting of homogeneous elements, then the ideal $I(S)$ is homogeneous and will be called the *algebraic ideal* generated by S. For any subset $S \subset \Omega^*(M)$ we denote by $\langle S \rangle$ the algebraic ideal generated by S. If S is the finite set $\theta^1, \theta^2, \dots, \theta^s$ of forms of arbitrary degrees, then we write

$$I = \langle S \rangle_{alg} = \langle \theta^1, \dots, \theta^s \rangle_{alg} = \langle \eta^1 \wedge \theta^1 + \dots + \eta^s \wedge \theta^s \, ; \, \eta^j \in \Omega^*(M) \rangle. \tag{3.2}$$

Definition 3.5: An ideal $I \subset \Omega^*(M)$ is said to be *closed* with respect to exterior differentiation if and only if the exterior derivative of any form in I is also in I (i.e. for all $\alpha \in I$ implies that $\boldsymbol{d}\alpha \in I$, or more compactly $\boldsymbol{d}I \subset I$). An algebraic ideal $I \subset \Omega^*(M)$ which is closed with respect to exterior differentiation is called a *differential ideal* on M. A differential ideal is also called an exterior differential system. For any subset $S \subset \Omega^*(M)$ we denote by $\langle S, \boldsymbol{d}S \rangle$ the differential ideal generated by S, which is the ideal generated by all the forms in S and their exterior derivatives. If S is the finite set $\theta^1, \theta^2, \dots, \theta^s$ of forms of arbitrary degrees, then the differential ideal generated by S is

$$I = \langle S \rangle_{\text{diff}} = \langle \theta^1, \dots, \theta^s \rangle_{\text{diff}} = \langle \theta^1, \dots, \theta^s ; \boldsymbol{d}\theta^1, \dots, \boldsymbol{d}\theta^s \rangle_{\text{alg}}$$

$$= \langle \eta^1 \wedge \theta^1 + \dots + \eta^s \wedge \theta^s + \gamma^1 \wedge \boldsymbol{d}\theta^1 + \dots + \gamma^s \wedge \boldsymbol{d}\theta^s ; \, \eta^i, \gamma^i \in \Omega^*(M) \rangle. \tag{3.3}$$

In particular, if the θ^r are 1-forms, then we will say that $I = \langle \theta^r, \boldsymbol{d}\theta^r \rangle$ is a Pfaffian differential ideal.

Definition 3.6: An *exterior differential system* (EDS) is a pair (M, I) where M is a smooth manifold and $I \subset \Omega^*(M)$ is a differential ideal in the ring $\Omega^*(M)$ of differential forms on M. An exterior differential system with independent condition on a manifold M is a pair (I, Ω), where I is an exterior differential ideal and Ω is a decomposable n-form such that $\Omega_x \notin I_x$ for all $x \in M$. this Ω, or its equivalence class $[\Omega]$ up to scale, is called the independent condition.

Definition 3.7: Let (M, I) be an exterior differential system spanned by forms $\theta^1, \theta^2, \dots, \theta^s$. An *integral manifold N* for the system is an immersed sub manifold $i: N \longrightarrow M$, which satisfies $i^*(\theta) = 0$, $\forall \theta \in I$. An integral manifold of an exterior differential system I on M with independent condition Ω is an immersed sub manifold $i: N \longrightarrow M$ such that $i^*(\theta) = 0$, $\forall \theta \in I$ and $i^*(\Omega) \neq 0$ at each point of M.

4. CARTAN THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, we introduce the relationship between differential equations and exterior differential system.

A system of partial differential equations, with any number of independent and dependent variables and involving partial derivatives of any order, can be written as an exterior differential systems.

Elie Cartan developed the theory of exterior differential systems as a coordinate-free way to describe and study partial differential equations. Cartan's geometric theory of partial differential equations essentially consists of the description of a PDE as a differential ideal I (equivalently, an exterior differential system) on a manifold M. Then an integral manifold of I will be the pair (N, f), where N is a manifold and $f: N \to M$ is an embedding such that $f^*(\omega) = 0$, for all ω in the Ideal I. We recall that systems of PDE′s can be represented as systems of homogeneous p-form equations (PDE′s in the form of exterior differential equations) by possibly introducing new variables. We describe M to be a manifold with local coordinates consisting of all independent and dependent variables in the PDE′s and auxiliary variables. Local coordinates of N will consist of independent variables. We then define the forms ω on M such that their restriction to N gives the PDE′s we started with.

Given a set of forms, $\alpha = a_i dx^i$, $\omega = a_{ij} dx^i \wedge dx^j$, etc, when restricted to, as those sub manifolds on which the forms, become identically zero. Cartan noted that such a problem amounts to finding solutions of a coupled set of first-order partial differential equations-the dimensionality n of the submanifold sought means that n independent variables, say x^i, can be introduced autonomously, and a set of linear homogeneous equations $\alpha = a_i dx^i = 0$ arises from restriction of each given 1-form, a set of homogeneous quadratic equations from each given 2-form $\omega = a_{ij} dx^i \wedge dx^j = 0$, etc.

If there are, say, s_o independent 1-forms in the given set, $\alpha^1, \alpha^2, \dots, \alpha^{s_o}$, any s_o linearly independent 1-forms formed from them would yield the same partial differential equations. If to a 2-form σ we had added any other, or combinations of terms such as $\theta \wedge \omega^1$ where θ is an arbitrary 1-form, again no essential change would occur in the homogeneous equations. In sum, any set of generators of an ideal I of forms is sufficient. It is the ideal I that geometrizes the partial differential equations. This is the essence of Cartan's approach. The partial differential problem, is then expressed in a geometrical form which we believe expedites both the conceptualization, and systematic local analysis, of many otherwise ad hoc "games" of applied mathematics.

Example 1: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \tag{4.1}$$

To phrase (4.1) geometrically, we introduce variables $x, y, u, p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy}$. Let M be a manifold with variables (x, y, u, p, q, r, s, t), on M we define the following 1-forms:

$$\left.\begin{aligned} \theta^1 &= du - pdx - qdy \\ \theta^2 &= dp - rdx - sdy \\ \theta^3 &= dq - sdx - tdy \end{aligned}\right\}, \tag{4.2}$$

And the partial differential equation (4.1) is then modeled by the differential ideal $I \subset \Omega^*(M)$ generated by the one-forms θ^1, θ^2 and θ^3. The solutions of the PDE (4.1) are in one-one correspondence with the integral manifolds of the exterior differential system $I = \langle \theta^1, \theta^2, \theta^3 \rangle$ with independence condition $\Omega = dx \wedge dy \neq 0$.

Example 2: Consider cars moving along a one-lane road. If cars are not created or destroyed , the conservation law of cars given by

$$\frac{\partial k}{\partial x} + \frac{\partial \rho}{\partial t} = 0, \tag{4.3}$$

Where $\rho(x, t)$ be the local density of cars and $k(x, t)$ the local flux of cars.

To phrase (4.3) in coordinate-free manner, we define the exterior differential system generated by the following 1-form

$$\alpha = -\rho dx + kdt, \tag{4.4}$$

With the independence condition $\Omega = dt \wedge dx \neq 0$.

By taking the exterior derivative of both sides of (4.4) we obtain

$$\begin{aligned} \boldsymbol{d}\alpha &= -\boldsymbol{d}\rho \wedge dx + \boldsymbol{d}k \wedge dt \\ &= -\left(\frac{\partial \rho}{\partial t} dt + \frac{\partial \rho}{\partial x} dx\right) dx + \left(\frac{\partial k}{\partial t} dt + \frac{\partial k}{\partial x} dx\right) dt. \end{aligned} \tag{4.5}$$

Using the properties of wedge product, this becomes

$$\boldsymbol{d}\alpha = -\left(\frac{\partial k}{\partial x} + \frac{\partial \rho}{\partial t}\right) dtdx. \tag{4.6}$$

Therefore, equation (4.3) turns into

$$\boldsymbol{d}\alpha = 0, \tag{4.7}$$

This is the coordinate-free version of conservation law of cars.

5. THE CONTINUITY EQUATION AS AN EXTERIOR DIFFERENTIAL SYSTEMS

In this section, we use the Cartan's theory to show that it is possible to rewrite the continuity equation as an exterior differential systems.

A. Compressible (general) case:

To reformulate the continuity equation (2.2) as an exterior differential system, we set

$$\left.\begin{aligned} \theta^1 &= dx^1 - v^1 dt \\ \theta^2 &= dx^2 - v^2 dt \\ \theta^3 &= dx^3 - v^3 dt \end{aligned}\right\}. \tag{5.1}$$

We define the 3-form:

$$\begin{aligned} \omega &= \rho(dx^1 - v^1 dt) \wedge (dx^2 - v^2 dt) \wedge (dx^3 - v^3 dt) \\ &= \rho[dx^1 dx^2 dx^3 - v^1 dx^2 dx^3 dt + v^2 dx^3 dt dx^1 - v^3 dt dx^1 dx^2]. \end{aligned} \tag{5.2}$$

By applying the exterior derivative of both sides in (5.2)

$$\begin{aligned} \boldsymbol{d}\omega &= \boldsymbol{d}[\rho dx^1 dx^2 dx^3 - \rho v^1 dx^2 dx^3 dt + \rho v^2 dx^3 dt dx^1 - \rho v^3 dt dx^1 dx^2] \\ &= \boldsymbol{d}\rho \wedge dx^1 dx^2 dx^3 - \boldsymbol{d}(\rho v^1) \wedge dx^2 dx^3 dt + \boldsymbol{d}(\rho v^2) \wedge dx^3 dt dx^1 - \boldsymbol{d}(\rho v^3) \wedge dt dx^1 dx^2 \\ &= \left(\frac{\partial \rho}{\partial t} dt + \frac{\partial \rho}{\partial x^1} dx^1 + \frac{\partial \rho}{\partial x^2} dx^2 + \frac{\partial \rho}{\partial x^3} dx^3\right) dx^1 dx^2 dx^3 \\ &\quad - \left(\frac{\partial \rho v^1}{\partial t} dt + \frac{\partial \rho v^1}{\partial x^1} dx^1 + \frac{\partial \rho v^1}{\partial x^2} dx^2 + \frac{\partial \rho v^1}{\partial x^3} dx^3\right) dx^2 dx^3 dt \\ &\quad + \left(\frac{\partial \rho v^2}{\partial t} dt + \frac{\partial \rho v^2}{\partial x^1} dx^1 + \frac{\partial \rho v^2}{\partial x^2} dx^2 + \frac{\partial \rho v^2}{\partial x^3} dx^3\right) dx^3 dt dx^1 \\ &\quad - \left(\frac{\partial \rho v^3}{\partial t} dt + \frac{\partial \rho v^3}{\partial x^1} dx^1 + \frac{\partial \rho v^3}{\partial x^2} dx^2 + \frac{\partial \rho v^3}{\partial x^3} dx^3\right) dt dx^1 dx^2. \end{aligned} \tag{5.3}$$

Using the properties of wedge product, this becomes

$$\begin{aligned} \boldsymbol{d}\omega &= \frac{\partial \rho}{\partial t} dt dx^1 dx^2 dx^3 - \frac{\partial \rho v^1}{\partial x^1} dx^1 dx^2 dx^3 dt + \frac{\partial \rho v^2}{\partial x^2} dx^2 dx^3 dt dx^1 - \frac{\partial \rho v^3}{\partial x^3} dx^3 dt dx^1 dx^2 \\ &= \left[\frac{\partial \rho}{\partial t} + \frac{\partial \rho v^1}{\partial x^1} + \frac{\partial \rho v^2}{\partial x^2} + \frac{\partial \rho v^3}{\partial x^3}\right] dt dx^1 dx^2 dx^3. \end{aligned} \tag{5.4}$$

Therefore $\boldsymbol{d}\omega = 0$ corresponds to

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v^1}{\partial x^1} + \frac{\partial \rho v^2}{\partial x^2} + \frac{\partial \rho v^3}{\partial x^3} = 0. \tag{5.5}$$

Notice that (5.5) is exactly the same as (2.2).Thus the continuity equation correspond to the closed form

$$\boldsymbol{d}\omega = 0, \tag{5.6}$$

This equation represents the coordinate-free form of the conservation of mass.

The property of conservation of mass (Equation (5.6)), is related to the fact that the 3-form ω is an integral-invariant [7] for the flow. By this we mean: Given two 3-chains $c_3, c_3' \in C_3(M)$ which are in $1-1$ correspondence such that corresponding points lie on the same trajectory of the flow $\{\phi_t\}$, then:

$$\int_{c_3} \omega = \int_{c_3'} \omega, \qquad c_3' = (\phi_t)_* c_3. \tag{5.7}$$

If now $c_3 = c_3^{(0)}$ at $t_0 = 0$, then, by (5.7),

$$\int_{c_3} \omega = \int_{c_3'} \omega \,|_{t=t_0} = \int_{c_3} \rho dx^1 dx^2 dx^3, \tag{5.8}$$

And following up $c_3^{(0)} = c_3^{(1)}$ at time t_1, we have:

$$\int_{c_3^{(0)}} \rho dx^1 dx^2 dx^3 = \int_{c_3^{(1)}} \rho dx^1 dx^2 dx^3, \tag{5.9}$$

Which expresses that mass is preserved in the flow $\{\phi_t\}$, another form of the conservation of mass.

B. Inviscid, incompressible, and irrotational case (Potential flow):

A potential flow describes what flow would be if it were inviscid, incompressible, and irrotational. The problem is described by Laplace equation (equation (2.9)).

In order to reformulate the problem in a differential geometry terms, the nature of φ needs to be established first. It is known that φ is a scalar function, and thus should be a 0-form or a 2-form.

To phrase the Laplace equation (2.9) geometrically, we introduce the following variables

$$p = \frac{\partial \varphi}{\partial x}, q = \frac{\partial \varphi}{\partial y}, r = \frac{\partial \varphi}{\partial z}. \quad (5.10)$$

Let M be a manifold with variables $(x, y, z, \varphi, p, q, r)$, on M we define the 1-form

$$\alpha = d\varphi - pdx - qdy - rdz, \quad (5.11)$$

And the 2-form

$$\sigma = pdydz + qdzdx + rdxdy. \quad (5.12)$$

Taking the exterior derivative of (5.12) we obtain

$$\begin{aligned} \boldsymbol{d}\sigma &= \boldsymbol{d}[pdydz + qdzdx + rdxdy] \\ &= \boldsymbol{d}p \wedge dydz + \boldsymbol{d}q \wedge dzdx + \boldsymbol{d}r \wedge dxdy \\ &= \left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)dydz + \left(\frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy + \frac{\partial q}{\partial z}dz\right)dzdx + \left(\frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz\right)dxdy. \quad (5.13) \end{aligned}$$

Using the anti-symmetric property (property ii) of wedge product, this becomes

$$\begin{aligned} \boldsymbol{d}\sigma &= \frac{\partial p}{\partial x}dxdydz + \frac{\partial q}{\partial y}dydzdx + \frac{\partial r}{\partial z}dzdxdy \\ &= \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}\right]dxdydz. \quad (5.14) \end{aligned}$$

Note that $\boldsymbol{d}\sigma = 0$ is the same as equation (2.9). Then Laplace equation is equivalents to the relation

$$\boldsymbol{d}\sigma = 0. \quad (5.15)$$

This is the coordinate-free version of (2.9).

6. CONCLUSION

In this paper, we presented the method of exterior differential systems for analyzing partial differential equations, the continuity equation and Laplace equation of fluid mechanics, are reformulated as an exterior differential systems.

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